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SENSITIVITY OF POST-CRITICAL STATES TO CHANGES IN DESIGN PARAMETERS

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Abstract—Sensitivity of post-buckling paths is studied in the context of the general theory of elastic stability of discrete structural systems. It is assumed that the sensitivity of the critical state itself has been computed and a formulation is developed to account for sensitivity of the curvature of the post-critical states when there are changes in design parameters. A linear fundamental path is considered. Explicit expressions are obtained for the sensitivities and they take the form of perturbation expansions. Only first order sensitivity of post-critical paths has been developed. A simple example of an angle section column with deformable cross-section illustrates that although a critical state may be insensitive to changes in certain design parameters, the post-critical response may be highly sensitive. In the first example presented (an axially loaded angle section column), the post-buckling response even changes from stable to unstable depending on the values of the design parameter considered. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

This paper presents, for the first time, a theory to account for changes in the post-buckling path when design parameters are modified. This is a new field in the theory of elastic stability: the design sensitivity of post-critical states. No previous analytical work is available in this area.

In sensitivity analysis, variations or derivatives of state fields are found due to variations in the design parameters. An excellent account of the work in this area is given in the book by Haug *et al.* (1985), with application to static, buckling and dynamic analysis. Sensitivity analysis is important in design improvements and optimization; in stochastic analysis [Kleiber and Hien (1992); Godoy (1995)] and it is also information of actual relevance in itself.

Sensitivity in buckling problems is a difficult topic, since it involves a problem that is non-linear. The critical loads are usually computed from an eigenvalue problem and there is by now an important amount of information in this field, notably the work of Mroz (1992). The adjoint method was presented by Dems and Mroz (1983), and later extended to sensitivity with respect to changes in shape and boundary conditions of the system [Mroz (1992)] and thermo-elastic problems [Dems (1987)]. Second order sensitivity using direct and adjoint approaches may be found in Godoy *et al.* (1994), and Godoy and Raichman (1995).

Buckling loads, as computed from a bifurcation analysis, provide only limited information on the mechanics of the problem, especially in shell and shell-like structures. In such cases, there is a need to obtain not just critical states but also post-critical states (i.e., post-critical equilibrium paths emerging from the critical state). Sensitivity of post-critical states to imperfections (specifically geometric or load imperfections) has been the subject of research for some time and is part of the general theory of elastic stability [see Thompson and Hunt (1973), Flores and Godoy (1992) and Godoy and Mook (1995)]. Sensitivity of post-buckling states to changes in design parameters, on the other hand, has only been explored by numerical experimentation [Fitch and Budianski (1970), Flores and Godoy (1991); and many others]. But there seems to be a complete lack of analytical studies oriented to obtain explicit forms for design sensitivity in post-buckling analysis.

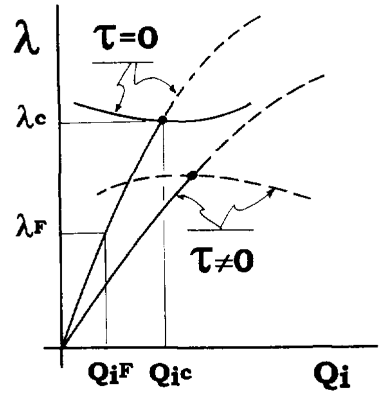

Fig. 1. Examples of primary and secondary paths for a reference system and for a system with changes in a design parameter.

To further clarify the problem studied, let us consider the response shown in Fig. 1. Equilibrium states are plotted in a load–displacement plane, assuming some displacement component Q_i and a single load parameter Λ. Solid lines represent stable equilibrium states, while dotted lines are associated to unstable states. The fundamental or primary path of equilibrium states arises from the origin and may in general be non-linear (in this paper, however, the presentation is restricted to linear fundamental paths). A critical state is also plotted and assumed to occur at Λ^c and Q_i^c. A non-linear path (known as secondary or post-critical path) emerges from that state. In Fig. 1, such a secondary path is plotted as a stable path, in the sense that there are equilibrium states at values higher than the critical load. All the above may be computed for a specific value of a design parameter τ, e.g. $\tau = 0$. But for $\tau \neq 0$ several changes may occur in the response: first, the primary path may change; second, the critical state may change; and finally, the post-critical path may be modified in values and also in nature (changing from stable to unstable).

This paper deals with a formulation for design sensitivity of post-critical equilibrium states. The framework of analysis is the so-called general theory of elastic stability, for discrete structural systems [Thompson and Hunt (1973)]. Use is made of perturbation analysis to find the critical and post-critical states for a reference configuration, and a summary of the relevant equations is presented in Section 2. Sensitivity of a critical state is the subject of Section 3, leading to explicit expressions for a linear fundamental path. Sensitivity of post-critical states is considered in Sections 4 and 5, for symmetric and asymmetric bifurcations, respectively. Finally, a simple example is presented in Section 6 to illustrate the procedure in a three degree-of-freedom problem.

2. SUMMARY OF POST-CRITICAL ANALYSIS

There are several formulations based on the total potential energy of the system, available for the analysis of post-critical behavior, notably the V-formulation, in which the original set of coordinates is used for both pre- and post-buckling analysis; the W-formulation, in which the energy is computed using sliding generalized coordinates measured from the fundamental path; and the D-formulation, in which the second variation is set to a diagonal form for each load level. We follow the first approach in this work, as originally presented in Flores and Godoy (1992).

The total potential energy V is written in terms of the generalized coordinates Q_i and the load parameter Λ, i.e. $V = V[Q_j, \Lambda]$. The equilibrium condition is equivalent to the condition that the total potential energy is at a stationary state, leading to

$$V_i[Q_i, \Lambda] = 0, \tag{1}$$

where subscripts of V indicate derivation with respect to Q_i. Solution of eqn (1) leads to the fundamental or primary equilibrium path $Q_i^F = Q_i(\Lambda^F)$.

A critical state associated to a distinct (not compound) critical mode is characterized by

$$V_{ij}x_j|^{c} = 0, \tag{2}$$

where x_j^c is the eigenvector, Λ^c is the eigenvalue and $V_{ij}^c = V_{ij}[Q_j^F, \Lambda^c]$. Thus, a critical state is characterized by the load and displacement at which it occurs (namely Λ^c and Q_i^c) and the direction of instability x_j^c.

The nature of the critical state is a bifurcation if

$$V_i' x_i|^{c} = 0. \tag{3}$$

A non-zero value in eqn (3) leads to a limit point. A symmetric bifurcation satisfies

$$C = V_{ijk}x_i x_j x_k \tag{4}$$

and asymmetric bifurcations are associated to $C \neq 0$.

The post-critical states are written in terms of a suitable perturbation parameter that will be denoted s. This parameter is usually taken as one of the components of the displacement vector, that has a non-zero value in the post-critical path. The post critical displacements and load become

$$Q_i(s) = Q_i^c + q_i(s)$$
$$\Lambda(s) = \Lambda^c + \lambda(s), \tag{5}$$

with

$$q_i(s) = q_i^{(1)c}s + \tfrac{1}{2}q_i^{(2)c}s^2 + \ldots$$
$$\lambda(s) = \lambda^{(1)c}s + \tfrac{1}{2}\lambda^{(2)c}s^2 + \ldots \tag{6}$$

The notation $(\ )^{(n)c}$ indicates the n-derivative with respect to the perturbation parameter s, evaluated at the critical state. To obtain the derivatives in eqn (6), use is made of the perturbation equations of equilibrium. Following Flores and Godoy (1992), the results from the perturbation analysis are as follows.

For symmetric bifurcation, the procedure to compute the unknown coefficients in eqn (6) may be stated as

$$\lambda^{(1)c} = 0 \tag{7}$$

$$q_j^{(1)c} = x_j \tag{8}$$

$$\text{solve } V_{ij}y_j = -V_i' \tag{9}$$

$$\text{solve } V_{ij}z_j = -V_{ijk}x_j x_k \tag{10}$$

$$\text{compute } B = (V_{ijk}y_k + V_{ij}')x_i x_j|^{c} \tag{11}$$

$$\text{compute } \tilde{V}_4 = V_{ijkl}x_i x_j x_k x_l + 3V_{ijk}x_i x_j z_k|^{c} \tag{12}$$

$$\text{calculate } \lambda^{(2)c} = -\frac{\tilde{V}_4}{3B} \tag{13}$$

$$\text{solve } V_{ij}q_j^{(2)} = -V_{ijk}. \tag{14}$$

For asymmetric bifurcation, the algorithm becomes

$$\text{solve } V_{ij}y_j|^c = -V_i' \tag{15}$$

$$\text{compute } A = V_{ijk}x_iy_jy_k + 2V_{ij}'x_iy_j + V_i''x_i|^c \tag{16}$$

$$\text{calculate } \lambda^{(1)c} = -\frac{B-\sqrt{B^2-AC}}{A} \tag{17}$$

$$q_j^{(1)} = x_j + \lambda^{(1)c}y_j. \tag{18}$$

The above procedures are similar to those obtained using the W-formulation and presented, e.g. in Thompson and Hunt (1973).

3. SENSITIVITY OF CRITICAL STATE

We shall see that sensitivity of the post-critical path depends on sensitivity of the critical (and also pre-critical) state. Thus, we review here some basics of first order sensitivity analysis of critical states, based on the work of Godoy *et al.* (1994).

Consider a design parameter τ, and we are interested in the sensitivity of the buckling problem when changes are introduced in τ. Let λ^c, Q_j^c and x_j^c be the variables defining the critical state, but for a given value of the design parameter τ, usually $\tau = 0$. If the evaluated derivatives with respect to τ are denoted by

$$(\dot{\ }) = \left.\frac{\mathrm{d}()}{\mathrm{d}\tau}\right|_{\tau=0}$$

and

$$(\ddot{\ }) = \left.\frac{\mathrm{d}^2()}{\mathrm{d}\tau^2}\right|_{\tau=0},$$

then one may write the variables at the critical state as a Taylor expansion about a state with $\tau = 0$ in the form

$$\lambda(\tau) = \lambda^c + \dot{\lambda}\tau + \tfrac{1}{2}\ddot{\lambda}\tau^2 + \ldots \tag{19}$$

$$x_j(\tau) = x_j^c + \dot{x}_j\tau + \tfrac{1}{2}\ddot{x}_j\tau^2 + \ldots \tag{20}$$

$$Q_j(\tau) = Q_j^c + \dot{Q}_j\tau + \tfrac{1}{2}\ddot{Q}_j\tau^2 + \ldots \tag{21}$$

To review the analysis, we shall restrict our attention to a case in which the fundamental path is linear. The energy V is a function of Q_i and λ, as stated at the beginning of Section 2. An explicit form of V contains such variables multiplied by coefficients. Following the notation of Croll and Walker (1972), we write

$$V[q_j, \Lambda] = A_0 + \lambda A_iQ_i + \frac{1}{2}A_{ij}Q_iQ_j + \frac{1}{3!}A_{ijk}Q_iQ_jQ_k + \frac{1}{4!}A_{ijkl}Q_iQ_jQ_kQ_l, \tag{22}$$

where A_0, A_i, A_{ij}, A_{ijk} and A_{ijkl} are coefficients. This explicit dependence of V with the control and response variables will make clear the dependence of all coefficients involved, on the design parameter τ.

The fundamental path is given by the equilibrium condition

$$V_i = A_i + A_{ij}Q_j = 0 \tag{23}$$

and the critical state by

$$(A_{ij} + \Lambda^c A_{ijk} Q_k^F) x_j^c = 0 \tag{24}$$

with $x_i^c = 1$.

The algorithm for the computation of the sensitivity parameters in eqns (20)–(22) is as follows:

(1) Solve $A_{ij}\dot{Q}_j = -(\dot{A}_{ij}Q_j^F + \dot{A}_i)$
[notice that there is no constraint on $\dot{Q}_j$ since A_{ij} is not a critical state].
(2) Compute $v_i' = (\dot{A}_{ij} + \Lambda \dot{A}_{ijk} Q_k^F) x_j^c$.
(3) Calculate $\dot{\lambda} = -(\lambda^c A_{ijk} x_j^c \dot{Q}_k + v_i') x_i^c$.
(4) Compute $g_i' = v_i' + \dot{\lambda}(A_{ijk}Q_k^F)x_j^c + (\lambda^c A_{ijk} x_j^c)\dot{Q}_k$.
(5) Solve $(A_{ij} + \lambda^c A_{ijk} Q_k^F) h_i' = -g_i'$.
(6) Since $\dot{x}_1 = \alpha_1 x_1^c + h_1'$ therefore $\alpha_1 = 1$.
(7) Compute $\dot{x}_i = x_i^c + h_i'$.

Second (and higher) order coefficients can be computed in this way; this procedure is fully explained in Godoy *et al.* (1994). What has been reviewed here is a direct method of analysis; another possibility would be the adjoint method [see for example, Dems and Mroz (1983)].

In this section, normalization of the eigenvector is carried out by setting one of the components of the eigenvector equal to one, i.e. $x_1^c = 1$. However, other possibilities of normalization exist and one of them (frequently employed in structural dynamics) is to set $(A_{ijk}Q_k^F)x_j^c x_i^c = 1$. We shall not pursue such analysis here because the main interest in this work is on the post-buckling response, rather than any dynamic feature of the structure.

4. SENSITIVITY OF POST-CRITICAL STATES: SYMMETRIC BIFURCATION

Let $\lambda^{(2)c}$ be the curvature of the post-buckling path, calculated from a stability analysis, for a reference value of the design parameter τ, i.e. $\tau = 0$. In the post-critical analysis, the sensitivity of the post buckling state may be written in terms of the design parameter τ as:

$$\lambda^{(2)}(\tau) = \lambda^{(2)c} + \dot{\lambda}^{(2)}\tau + \tfrac{1}{2}\ddot{\lambda}^{(2)}\tau^2 + \dots$$

$$q_j^{(2)}(\tau) = q_j^{(2)c} + \dot{q}_j^{(2)}\tau + \tfrac{1}{2}\ddot{q}_j^{(2)}\tau^2 + \dots,$$

where, again, dots on top of a variable indicate derivation with respect to τ. Notice that $\dot{q}_i^{(1)} = \dot{x}_i$ and $\dot{x}_i$ was already computed. To obtain the coefficients in the previous equation, let us write eqn (13) in the form

$$3B\lambda^{(2)} + \tilde{V}_4|^c = 0. \tag{25}$$

First order perturbation of this equation requires

$$3B\dot{\lambda}^{(2)}|^c = -3\dot{B}\lambda^{(2)} - \dot{\tilde{V}}_4|^c. \tag{26}$$

To solve $\dot{\lambda}^{(2)}$ it is necessary to obtain $\dot{B}$ and $\dot{\tilde{V}}_4$, and this will be done in the following. Consider from eqns (11) and (22)

$$\tilde{V}_4 = A_{ijkl}x_i x_j x_k x_l + 3(A_{ijk} + A_{ijkl}Q_l^{\mathrm{F}})x_i x_j z_k$$

$$B = (A_{ijk} + A_{ijkl}Q_l^{\mathrm{F}})x_i x_j y_k.$$

The first order perturbation equations of $\tilde{V}_4$, called $\dot{\tilde{V}}_4$, is

$$\dot{\tilde{V}}_4 = (\dot{A}_{ijk}x_l + 4A_{ijkl}\dot{x}_l)x_i x_j x_k + 3\eta_{ik}x_i z_k + 3[(A_{ijk} + A_{ijkl}Q_l^{\mathrm{F}})x_i x_j]\dot{z}_k, \tag{27}$$

where

$$\eta_{ik} = (\dot{A}_{ijk} + \dot{A}_{ijkl}Q_l^{\mathrm{F}} + A_{ijkl}\dot{Q}_l)x_j + 2(A_{ijk} + A_{ijkl}Q_l^{\mathrm{F}})\dot{x}_j. \tag{28}$$

The first order perturbation equation of B is

$$\dot{B} = \eta_{ik}y_k x_i + [(A_{ijk} + A_{ijkl}Q_l^{\mathrm{F}})x_i x_j]\dot{y}_k. \tag{29}$$

We now need the derivatives $\dot{z}_k$ and $\dot{y}_k$. They are obtained from derivation of eqns (9) and (10) as follows:

$$(A_{ij} + \lambda^{\mathrm{c}}A_{ijk}Q_k^{\mathrm{F}})\dot{y}_j = -(\gamma_{ij}y_j + \dot{A}_i), \tag{30}$$

where

$$\gamma_{ij} = (\dot{A}_{ij} + \dot{\lambda}A_{ijk}Q_k^{\mathrm{F}} + \lambda\dot{A}_{ijk}Q_k^{\mathrm{F}} + \lambda A_{ijk}\dot{Q}_k) \tag{31}$$

and

$$(A_{ij} + \lambda^{\mathrm{c}}A_{ijk}Q_k^{\mathrm{F}})\dot{z}_j = -(\gamma_{ij}z_j + \eta_{ik}x_k). \tag{32}$$

At this stage, we need to set some values of $\dot{z}_1$ and $\dot{y}_1$ to be consistent with our earlier normalization of eigenvectors. Since we have already adopted $y_1 = 0$ and $z_1 = 0$, then it follows that $\dot{z}_1 = 0$ and $\dot{y}_1 = 0$.

An algorithm for computation of $\lambda^{(2)}$ for symmetric bifurcation is as follows.

(1) Compute

$$\gamma_{ij} = (\dot{A}_{ij} + \dot{\lambda}A_{ijk}Q_k^{\mathrm{F}} + \lambda\dot{A}_{ijk}Q_k^{\mathrm{F}} + \lambda A_{ijk}\dot{Q}_k)$$

$$\eta_{ik} = (\dot{A}_{ijk} + \dot{A}_{ijkl}Q_l^{\mathrm{F}} + A_{ijkl}\dot{Q}_l)x_j + 2(A_{ijk} + A_{ijkl}Q_l^{\mathrm{F}})\dot{x}_j.$$

(2) Solve $\dot{y}_i$ using $\dot{y}_1 = 0$ and

$$(A_{ij} + \lambda^{\mathrm{c}}A_{ijk}Q_k^{\mathrm{F}})\dot{y}_j = -(\gamma_{ij}y_j + \dot{A}_i).$$

(3) Solve $\dot{z}_j$ using $\dot{z}_1 = 0$ and

$$(A_{ij} + \lambda^{\mathrm{c}}A_{ijk}Q_k^{\mathrm{F}})\dot{z}_j = -(\gamma_{ij}z_j + \eta_{ik}x_k).$$

(4) Compute

$$a_{ijk} = A_{ijk} + A_{ijkl}Q_l^{\mathrm{F}}\dot{B} = \eta_{ik}x_i y_k + a_{ijk}x_i x_j \dot{y}_k.$$

(5) Compute

$$\dot{\tilde{V}}_4 = (\dot{A}_{ijkl}x_l + 4A_{ijkl}\dot{x}_l)x_i x_j x_k + 3\eta_{ik}x_i z_k + 3a_{ijk}x_i x_j \dot{z}_k.$$

(6) Evaluate

$$\dot{\lambda}^{(2)} = -\frac{3\dot{B}\lambda^{(2)c} - \dot{\tilde{V}}_4}{3B}.$$

It may also be required to compute the sensitivity of post-critical displacements. To do that, let us obtain the derivatives of eqn (14)

$$(\dot{A}_{ij} + \dot{\lambda} A_{ijk} Q_k + \lambda \dot{A}_{ijk} Q_k + \lambda A_{ijk} \dot{Q}_k) q_j^{(2)} + (A_{ij} + \lambda A_{ijk} Q_k) \dot{q}_j^{(2)}$$
$$= -[(\dot{A}_{ijk} + \dot{A}_{ijkl} Q_l + A_{ijkl} \dot{Q}_l) x_j x_k + 2(A_{ijk} + A_{ijkl} Q_l) \dot{x}_i x_j + \dot{\lambda}^{(2)} A_i + \lambda^{(2)c} \dot{A}_i]$$

and now $\dot{q}_i^{(2)}$ can be obtained by solving the system

$$(A_{ij} + \lambda A_{ijk} Q_k) \dot{q}_i^{(2)} = -(\gamma_{ij} \dot{q}_j^{(2)} + \eta_{ik} x_k + \dot{\lambda}^{(2)} A_i + \lambda^{(2)c} \dot{A}_i). \tag{33}$$

A similar scheme of analysis can be followed to compute the second order sensitivities of the post-buckling curvature $\ddot{\lambda}^{(2)c}$.

5. SENSITIVITY OF POST-CRITICAL STATES: ASYMMETRIC BIFURCATION

The coefficients of the perturbation expansion in this case arise from eqn (17), i.e. $A\lambda^{(1)c} = -B + [B^2 - AC]^{1/2}$. First order perturbation of this equation leads to

$$\dot{A}\lambda^{(1)c} + A\dot{\lambda}^{(1)} = -\dot{B} + \tfrac{1}{2}[B^2 - AC]^{-1/2}(2\dot{B}B - \dot{A}C - A\dot{C}). \tag{34}$$

The derivative $\dot{B}$ was obtained previously in eqn (29). We need to calculate $\dot{A}$ and $\dot{C}$ from the derivatives of eqns (16) and (4), now in the form

$$C = a_{ijk} x_i x_j x_k |^{c} \tag{35}$$

$$A = a_{ijk} x_i y_j y_k |^{c}, \tag{36}$$

where $a_{ijk} = (A_{ijk} + A_{ijkl} Q_l^{F})$. The first order derivatives are

$$\dot{C} = (\dot{A}_{ijk} + \dot{A}_{ijkl} Q_l^{F} + A_{ijkl} \dot{Q}_l) x_i x_j x_k + 3 a_{ijk} \dot{x}_i x_j x_k \tag{37}$$

$$\dot{A} = (\dot{A}_{ijk} + \dot{A}_{ijkl} Q_l^{F} + A_{ijkl} \dot{Q}_l) x_i y_j y_k + a_{ijk} (\dot{x}_i y_j y_k + 2 x_i \dot{y}_j y_k). \tag{38}$$

The computation of $\dot{y}_j$ is as in symmetric bifurcation [see eqn (30)]. From eqn (34) one may obtain

$$\dot{\lambda}^{(1)} = \frac{-\dot{B} + \frac{1}{2}[B^2 - AC]^{-1/2}(2\dot{B}B - \dot{A}C - A\dot{C}) - \dot{A}\lambda^{(1)c}}{A}. \tag{39}$$

An algorithm for the computation of $\dot{\lambda}^{(1)}$ in asymmetric bifurcation could proceed as follows:

(1) Compute $\gamma_{ij} = (\dot{A}_{ij} + \dot{\lambda} A_{ijk} Q_k^{F} + \lambda \dot{A}_{ijk} Q_k^{F} + \lambda A_{ijk} \dot{Q}_k)$ and $\eta_{ik} = (\dot{A}_{ijk} + \dot{A}_{ijkl} Q_l^{F} + A_{ijkl} \dot{Q}_l) x_j + 2(A_{ijk} + A_{ijkl} Q_l^{F}) \dot{x}_j$.
(2) Solve $\dot{y}_i$ using $\dot{y}_i = 0$ and

$$(A_{ij}+\lambda^{c}A_{ijk}Q_{k}^{F})\dot{y}_{j} = -(\gamma_{ij}y_{j}+\dot{A}_{i}).$$

(3) Compute

$$\dot{A} = \eta_{ij}y_{j}y_{k}+a_{ijk}(-\dot{x}_{i}y_{j}y_{k}+2x_{i}\dot{y}_{j}y_{k})$$

$$\dot{B} = \eta_{ik}x_{i}y_{k}+a_{ijk}x_{i}x_{j}\dot{y}_{k}$$

$$\dot{C} = \eta_{ik}x_{i}x_{k}+a_{ijk}\dot{x}_{i}x_{j}x_{k}.$$

(4) Evaluate

$$\dot{\lambda}^{(1)} = \frac{-\dot{B}+\frac{1}{2}[B^{2}-AC]^{-1/2}(2\dot{B}B-\dot{A}C-A\dot{C})-\dot{A}\lambda^{(1)c}}{A}.$$

Notice that

- Only first order sensitivity of the critical state is required to solve first order sensitivity of the post-critical state.
- We have to solve two systems of linear equations for $\dot{y}$ and $\dot{z}$, and the rest are "just" products.

6. EXAMPLE OF SENSITIVITY OF POST-BUCKLING PATH

6.1. *Angle section column with deformable cross-section*

An academic, but interesting, example of sensitivity is the post-buckling response of an axially loaded, simply supported column with deformable cross-section. The cross-section considered is shown in Fig. 2; this is a model of an angle section column in which the two plates are connected by a hinge and a moment spring, of stiffness K. This example is considered in Eterovic *et al.* (1990), and more recently, in Lopez-Anido and Godoy (1995).

The axial (u) and transverse (w) displacements, and the rotation at the hinge (θ), are represented by

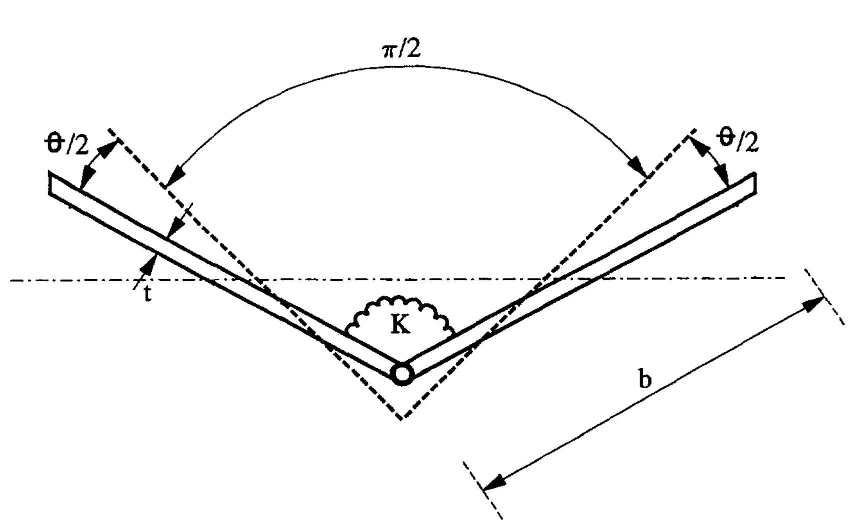

Fig. 2. Geometry of the cross-section of an axially loaded column.

$$u(x) = Q_1 \frac{x}{L}$$
$$w(x) = Q_2 \sin\left(\frac{\pi x}{L}\right)$$
$$\theta(x) = Q_3 \sin\left(\frac{\pi x}{L}\right), \tag{40}$$

where Q_1, Q_2 and Q_3 are the amplitudes of the assumed shape of u, w and θ. We use non-linear kinematic relations for column and linear constitutive equations, with a moment of inertia defined as

$$I(\theta) = I_0\left[1-\left(\theta-\frac{\theta^3}{6}\right)\right], \tag{41}$$

where I_0 is the moment of inertia of the undeformed section. It is assumed that the moment of inertia about the weak axis decreases with deformations of the cross section, θ.

The coefficients for the column under axial load, with a deformable cross section, are [Lopez-Anido and Godoy (1995)]

$$A_1 = -\Lambda L$$
$$A'_1 = -L$$
$$A_{11} = L$$
$$A_{22} = \frac{\pi^4}{2} dL$$
$$A_{33} = \tfrac{1}{2} fL$$
$$A_{122} = \frac{\pi^2}{2} L$$
$$A_{223} = -\tfrac{4}{3}\pi^3 dL$$
$$A_{2222} = \tfrac{9}{8}\pi^4 L, \tag{42}$$

where

$$\Lambda \equiv \frac{P}{EA_0}; \quad d \equiv \frac{I_0}{A_0 L^2}; \quad f \equiv \frac{K}{EA_0}.$$

The following results are obtained by use of the theory of elastic stability on the present model of the column, and yields the fundamental path, critical state and post-critical path.

The fundamental path is given by

$$Q^F = \begin{Bmatrix} \Lambda \\ 0 \\ 0 \end{Bmatrix}.$$

The critical state is given by

$$\Lambda^c = -\pi^2 d$$

$$Q^c = \begin{Bmatrix} -\pi^2 d \\ 0 \\ 0 \end{Bmatrix}$$

$$x^c = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}.$$

The load was initially assumed as tensile in writing V; for that reason, the critical load is negative. The perturbation parameter adopted to follow the post-critical path is the component Q_2. The vectors required for the post-buckling path are

$$y = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$z = \begin{Bmatrix} -\frac{\pi^2}{2} \\ 0 \\ \frac{8}{3}\pi^3 \frac{d}{f} \end{Bmatrix}.$$

The stability coefficient and B are

$$\tilde{V}_4^c = \tfrac{3}{8}\pi^4 L\xi$$

$$\xi = 1 - \tfrac{159}{9}\pi^2 \frac{d^2}{f}$$

$$B = V_{122} = \tfrac{1}{2}\pi^2 L.$$

Finally, the second derivative of the load is

$$\lambda^{(2)c} = -\left(\frac{\pi}{2}\right)^2 \xi.$$

Notice that a positive value of ξ indicates a curvature with the same sign as the critical load, thus a rising path.

6.1.1. *Sensitivity analysis*. Let us consider the design parameter K, the stiffness of the moment spring, and write it in the parametric form

$$K = K_0(1+\tau).$$

For convenience in the calculations, we shall assume a reference value K_0 such that $\xi = 0$. This means that

$$K_0 = \tfrac{256}{9}\pi^2 d^2 EA_0.$$

In the first stage, we make use of the sensitivity analysis of critical states outlined in Section 3. To obtain that, the derivatives of the energy coefficients with respect to τ are required; but since only A_{33} is a function of f (and thus of K and τ), the only non-zero derivative is

$$\dot{A}_{33} = \frac{1}{2}\frac{LK_0}{EA_0}.$$

It is simple to show that the fundamental path and critical state are not sensitive to changes in τ. Thus

$$\dot{Q}_j = 0; \quad \dot{x}_j = 0; \quad \dot{\lambda} = 0.$$

Next, we want sensitivity of $\dot{\lambda}^{(2)}$, for a problem of symmetric bifurcation. Since $\dot{A}_i = 0 = \dot{A}_{ijk}$ and $\dot{\lambda} = 0 = \dot{Q}_k$, then $\gamma_{ij} = \dot{A}_{ij}$ and the only non-zero term becomes $\gamma_{33} = \dot{A}_{33}$. Furthermore, $\eta_{ik} = 0$, leads to $\dot{y}_j = 0$ and $\psi_{33}y_3 = 0$.

To calculate the sensitivity of z we notice that

$$\psi_{33}z_3 = \dot{A}_{33}z_3 = \tfrac{4}{3}\pi^3 Ld.$$

The following system of equations should be solved, in which $\dot{z}_2 = 0$ (because the perturbation parameter adopted in this example is Q_2):

$$\begin{bmatrix} L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}fL \end{bmatrix} \begin{Bmatrix} \dot{z}_1 \\ 0 \\ \dot{z}_3 \end{Bmatrix} = -\tfrac{4}{3}\pi^3 Ld \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

The solution of this system is

$$\dot{z} = -\tfrac{8}{3}\pi^3 \frac{EA_0}{K} d \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

The sensitivity of B reduces to:

$$\dot{B} = (A_{ijk} + A_{ijkl}Q_l^{\mathrm{F}})x_i x_j \dot{y}_k = 0$$

Finally, the sensitivity $\dot{V}_4$ may be calculated in the form:

$$\begin{aligned} \dot{V}_4 &= 3(A_{ijk} + A_{ijkl}\tilde{Q}_l)x_i x_j \dot{z}_k \\ &= 3(A_{223})x_2 x_2 \dot{z}_3 \\ &= (-4\pi^3 dL)\left(-\tfrac{8}{3}\pi^3 \frac{EA_0}{K_0} d\right) \\ &= \tfrac{32}{3}\pi^6 d^2 \frac{EA_0 L}{K_0}. \end{aligned}$$

With the above results, it is possible to proceed with sensitivity of $\dot{\lambda}^{(2)c}$:

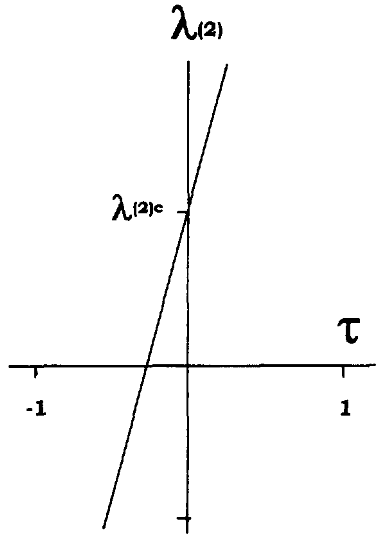

Fig. 3. Sensitivity of the curvature of the post-buckling path for an angle section column.

$$\dot{\lambda}^{(2)} = -\frac{\dot{V}_4}{3B} = -\tfrac{64}{9}\pi^4 d^2 \frac{EA_0}{K_0}.$$

The final result for sensitivity of the curvature of the post-buckling path is

$$\lambda^{(2)}(\tau) = \lambda^{(2)c} - \left(\frac{64}{9}\pi^4 d^2 \frac{EA_0}{K_0}\right)\tau + \ldots$$

The results are presented in Fig. 3 and it is seen there that first order sensitivity of the curvature of the post-buckling path changes sign with the value of τ. The system is an unstable symmetric bifurcation with negative values of τ; while for positive values of τ the system is a stable symmetric bifurcation.

For this simple example, the values of sensitivity could have been obtained from the explicit expressions; however, in more complex problems involving, e.g. finite element discretizations, there are no analytical solutions available and the present analysis would provide new information on sensitivity.

6.2. *Circular plate under in-plane loading*

The buckling of circular plates has been considered by several authors; the sensitivity of the critical state has been solved by Godoy *et al.* (1994) and will be further studied here to extend the analysis to sensitivity of post-critical behavior. This case is more complex than the previous one, in the sense that all pre-critical, critical and post-critical states are sensitive to changes in the design parameter (the thickness of the plate).

The axial (u) and transverse (w) displacements are represented by

$$u = \frac{r}{R}Q_2; \quad w = Q_1 \cos\left(\frac{\pi r}{2R}\right), \tag{43}$$

where Q_1 and Q_2 are the amplitudes of the assumed shapes. The energy coefficients in this case result in

$$A'_2 = -2R$$

$$A_{11} = \frac{\pi^2}{2}(1.191+\nu)\frac{D}{R}$$

$$A_{22} = 2c(1+\nu)$$

$$A_{112} = 1.7337(1+\nu)\frac{c}{R}A_{1111} = 10.55\frac{c}{R^2} \tag{44}$$

where

$$c = \frac{Eh}{1-\nu^2}; \quad D = \frac{Eh^3}{12(1-\nu^2)}.$$

The fundamental path is given by

$$Q^{\mathrm{F}} = \begin{Bmatrix} 0 \\ -\dfrac{A'_2}{A_{22}} \end{Bmatrix}.$$

The critical state can be computed as

$$\Lambda^{\mathrm{c}} = \frac{A_{11}A_{22}}{A'_2 A_{112}}$$

$$Q^{\mathrm{c}} = \begin{Bmatrix} 0 \\ -\dfrac{A_{11}}{A_{112}} \end{Bmatrix}$$

$$x^{\mathrm{c}} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}.$$

The vectors required for the post-buckling path are:

$$y = \begin{Bmatrix} 0 \\ -\dfrac{A'_2}{A_{22}} \end{Bmatrix}$$

$$z = \begin{Bmatrix} 0 \\ -\dfrac{A_{112}}{A_{22}} \end{Bmatrix}.$$

The coefficients of the quadratic equation are

$$C = 0; \quad B = 1.7337.$$

The stability coefficient is in this case

$$\tilde{V}_4 = A_{1111} + 3A_{112}z_2$$

and the curvature of the post-buckling path becomes

$$\lambda^{(2)c} = \frac{A_{1111} - 3\dfrac{(A_{112})^2}{A_{22}}}{3A_{11}}.$$

6.2.1. *Sensitivity analysis*. Let us consider the thickness h as a design parameter and write it in the form

$$h = h_0(1+\tau).$$

The derivatives of the energy coefficients become

$$\dot{A}'_2 = 0$$

$$\dot{A}_{11} = 3A_{11}$$

$$\dot{A}_{22} = A_{22}$$

$$\dot{A}_{112} = A_{112}$$

$$\dot{A}_{1111} = A_{1111}$$

$$\dot{f}_i = 0. \tag{45}$$

Sensitivity of the fundamental path is given by

$$\dot{Q}_j = \begin{Bmatrix} 0 \\ -Q_2^F \end{Bmatrix}$$

and sensitivity of the critical state results in

$$\dot{\lambda} = 3\Lambda^c \dot{x}_j = 0.$$

Next, we calculate sensitivity of the post-critical path. The matrices γ and η have coefficients equal to zero, except for

$$\gamma_{11}; \quad \gamma_{22} = \dot{A}_{22}; \quad \eta_{12} = \dot{A}_{112} x_1.$$

This leads to

$$\dot{y} = \begin{Bmatrix} 0 \\ -\dfrac{\gamma_{22} y_2}{A_{22}} \end{Bmatrix}$$

and

$$\dot{z} = \begin{Bmatrix} 0 \\ -\dfrac{\gamma_{22} y_2 + \eta_{12}}{A_{22}} \end{Bmatrix}.$$

The sensitivity of B becomes zero. The derivative of the stability coefficient with respect to τ becomes

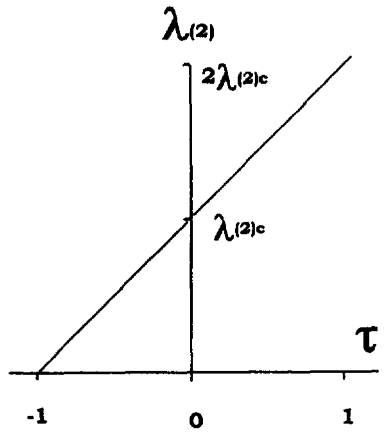

Fig. 4. Sensitivity of the curvature of the post-buckling path for a circular plate.

$$\dot{\tilde{V}}_4 = \dot{A}_{1111} + 3(\eta_{12} z_2 + A_{112} \dot{z}_2).$$

Finally, we can get

$$\dot{\lambda}^{(2)} = \lambda^{(2)}.$$

The result for sensitivity of the post-buckling curvature is

$$\lambda^{(2)}(\tau) = \lambda^{(2)c}(1+\tau)$$

and is plotted in Fig. 4.

7. CONCLUSIONS

A consistent derivation of first order sensitivity of the post-buckling path in symmetric as well as asymmetric bifurcation has been presented. The limitations of this work refer to a single design parameter and first order sensitivity. Certainly, both limitations could be overcome, but the author believes that this may be a good start to appreciate the difficulties and achievements in this field.

Once the information of design sensitivity of the critical state is obtained, following the present analysis it is possible to compute sensitivities of the curvature of the post-buckling path (in symmetric bifurcation) and sensitivity of the tangent to the post-buckling path (in asymmetric bifurcation). Algorithms for the computation of sensitivities are presented in both cases.

A three degrees-of-freedom problem has been first studied to illustrate the algorithms of computation. The problem is an axially loaded column in which the angle cross-section is assumed to rotate about a hinge that articulates both plates. The design parameter studied is the stiffness of a moment spring at the hinge. The results show that the critical state itself is not sensitive to the parameter considered; however, the post-buckling path is highly dependent on the stiffness coefficient adopted. Sensitivity is also reflected in that the post-buckling behavior may change from stable to unstable depending on the values of the design parameter.

In the second example, the circular plate under in-plane loading, we notice that the fundamental path and critical state are sensitive to changes in the design parameter chosen, and that the post-buckling path is also sensitive to thickness changes. However, in this case there cannot be a change from stable to unstable behavior produced by changes in the design parameter.

This is a new field in the context of design sensitivity analysis and it is expected that it will be further developed in the next few years. The present formulation would have more practical relevance if it was implemented in the context of a finite element package and this is seen as an important topic for further research.

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